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DESIGNING WORKED EXAMPLES TO SUPPORT MATHEMATICAL REASONING IN LEARNING CIRCLE CONCEPTS

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ABSTRACT

Students' reliance on routine procedures and difficulties in understanding abstract geometric concepts highlight the need for instructional designs that support mathematical reasoning. This study aims to construct a worked example design framework that facilitates students' transition from procedural understanding to mathematical reasoning in learning circles. The study employs a design and development research approach by adapting the Analysis, Design, and Development phases of the ADDIE model. the proposed worked example prototype was validated by two mathematics education experts using an expert validation sheet covering the aspects of material, construction, and language. The design integrates principles of Cognitive Load Theory and indicators of mathematical reasoning to produce a worked example prototype. The development process resulted in a worked example prototype that was validated by two mathematics education experts and deemed feasible for use after revisions. The prototype integrates Cognitive Load Theory principles and mathematical reasoning indicators to support students in recognizing patterns, formulating conjectures, and constructing logical arguments. This is achieved through an integrated presentation that minimizes extraneous cognitive load and the use of self-explanation prompts to encourage active cognitive processing. In addition, structured solution steps and paired problems are used as fading guidance to support the transition from guided learning to independent problem solving. Therefore, this study provides a worked example prototype that can serve as a reference for designing instructional materials to support mathematical reasoning in geometry learning.

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1. INTRODUCTION

Geometry is an essential branch of mathematics in developing students' spatial, logical, and abstract thinking skills. Jablonski & Ludwig (2023) state that geometry plays a central role in the curriculum as it supports visualization and mathematical problem-solving. Almubarak et al., (2025) emphasize that geometry learning strategies continue to evolve to enhance 21st-century spatial reasoning, while Crompton & Ferguson (2024) highlight that understanding geometric concepts forms a critical foundation for international learning standards. Instructional innovations such as gamification have also been implemented to increase student engagement in learning geometry (Kamalodeen et al., 2021). In particular, the concept of the circle has broad relevance in everyday life and across various scientific fields, making it a key component in fostering meaningful mathematical literacy and reasoning.

However, geometry instruction in schools is still largely dominated by procedural approaches and formula memorization, resulting in suboptimal development of students' conceptual understanding(Jablonski & Ludwig, 2023)argue that teaching tends to focus more on final results rather than deep thinking processes. Crompton & Ferguson (2024) found that students struggle to connect geometric concepts with broader representations and reasoning. Furthermore, Weigand et al. (2025) indicate that geometry learning remains fragmented and lacks connections between concepts. Gregersen (2024) also reveals that although digital tools support exploration, students still face difficulties in providing conceptual justifications for geometric phenomena. The low performance in international assessments such as TIMSS and PISA further reinforces the indication of weak global understanding and reasoning in geometry.

Mathematical reasoning is one of the key process standards in the Principles and Standards for School Mathematics (NCTM, 2000), as it enables students to construct and evaluate mathematical arguments. However, numerous studies indicate that this ability remains low. Öz & Işık, (2024) found that most secondary students rely more on algorithmic reasoning than on creative reasoning. Hjelte et al. (2020)also show that mathematics education research still focuses more on procedures rather than conceptual reasoning. Lithner (2017) emphasizes that the dominance of imitative reasoning limits students' creativity, while Mukuka et al., (2023) reveal that teaching practices have not fully stimulated exploration and justification. These conditions highlight the need for instructional designs that systematically promote students' mathematical reasoning abilities.

One potential strategy is the use of worked examples, which guide students to understand the conceptual structure underlying problem-solving rather than merely following mechanical procedures (Sweller, 2020). Based on Cognitive Load Theory, explicitly presenting solution steps helps students allocate cognitive resources to analyze relationships between concepts and mathematical principles. Paas & van Merriënboer, (2020); Renkl, (2014) assert that through observing examples and engaging in reflection, students can construct knowledge schemas that support reasoning and mathematical generalization. Chen et al. (2023) demonstrate that this strategy strengthens understanding of solution steps and improves transfer to new situations. Thus, worked examples not only function as procedural guidance but also as a means to develop mathematical reasoning.

In geometry learning, structured worked examples help students trace mathematical reasoning processes and understand the logic behind each solution step. Bokosmaty et al., (2015) show that this strategy effectively reduces cognitive load while enhancing conceptual understanding of relationships among geometric elements. Hänze & Leiss (2022) further indicate that heuristic worked examples encourage students to connect procedures with real-life situations and engage in reflective reasoning. Additionally, the use of examples and generalization plays a significant role in building logical mathematical arguments (Jablonski & Ludwig, 2023). However, the effectiveness of this strategy largely depends on how instructional design explicitly facilitates reasoning processes.

The concept of the circle requires a deep understanding of relationships among elements such as radius, diameter, arc, and central angle, making it abstract and complex for students. Studies show that students often struggle to reason about these interrelationships, particularly when determining connections between area, circumference, and central angles (Demir et al., 2023). (Lehmann, 2024) argues that understanding the area of a circle requires mathematical argumentation and justification, not merely formula application. Similarly, Payadnya et al. (2023) demonstrate that learning trajectories based on reflective questions (what-if questions) can support students in developing generalization and higher-order thinking. Therefore, the topic of circles provides an appropriate context for implementing reasoning-based worked examples.

Various studies have demonstrated the effectiveness of worked examples in improving learning performance and monitoring in algebra, arithmetic, and general geometry topics (Barbieri & Booth, 2020; Barbieri et al., 2023;Lee & Chen, 2015; Baars et al., 2014;Pachman et al., 2014. Despite this, most studies position worked examples primarily as tools for accelerating procedural skill acquisition or correcting errors. Only limited research has explored how the structure of worked examples can be designed to stimulate germane cognitive load in order to foster adaptive reasoning. Particularly in the topic of circles, which involves complex spatial relationships, there is a lack of design frameworks that explicitly guide students' transition from procedural mimicry to mathematical argumentation. Consequently, the potential of worked examples as scaffolding for higher-order mathematical reasoning has not been fully optimized.

The absence of such design guidance is critical, as poorly structured worked examples may risk suppressing students' initiative to think independently. Therefore, an instructional design approach is needed that not only presents solution steps but also cognitively engages students in constructing explanations behind each step. The perspective of Cognitive Load Theory (CLT) serves as a crucial foundation in this design, particularly in managing the interaction between visual and textual elements so that students' working memory capacity can be allocated to reasoning activities such as recognizing patterns and forming generalizations, rather than merely performing arithmetic calculations.

In response to these theoretical and practical gaps, this study aims to construct a worked example design framework oriented toward facilitating students' mathematical reasoning in the topic of circles. Unlike previous studies that primarily examined worked examples as instructional tools for improving procedural performance, this study focuses on developing a design framework that explicitly integrates Cognitive Load Theory principles with mathematical reasoning indicators to support students' transition from procedural understanding to mathematical reasoning. This emphasis on the instructional design rationale, rather than on evaluating the effectiveness of the final product, constitutes the novelty of the study. It elaborates on how CLT principles, such as reducing the split-attention effect and incorporating self-explanation prompts, are integrated into problem structures to stimulate indicators of mathematical reasoning. The findings of this study are expected to provide a solid theoretical blueprint for researchers and educators in developing reasoning-oriented geometry instructional materials in the future. Accordingly, this study addresses the following research question: How can a worked example design framework integrating Cognitive Load Theory principles and mathematical reasoning indicators be developed to facilitate students' mathematical reasoning in learning circle concepts?

2. METHOD

This study employs a design and development research approach using the ADDIE development model (Analyze, Design, Development, Implementation, Evaluation). However, this study focuses on three phases of the ADDIE model, namely Analyze, Design, and Development, to design and develop a product in the form of worked example-based mathematics problems that facilitate students' mathematical reasoning abilities in the topic of circles.

In the analysis phase, the researcher identified relevant needs prior to designing and developing the mathematical tasks. This phase involved analyzing various literature and learning resources to identify students' difficulties and the urgency of using worked example-based tasks in enhancing mathematical reasoning in circle learning.

In the design phase, the researcher developed a product in the form of worked example-based mathematical problems. The designed tasks were aligned with indicators of mathematical reasoning and structured according to solution steps that students would follow, thereby supporting gradual conceptual understanding.

In the development phase, the researcher constructed the product based on the established design. The prototype of the tasks was developed through problem reconstruction and the organization of systematic solution steps. After the prototype had been developed, it was validated by two mathematics education lecturers from Universitas Negeri Yogyakarta (UNY) with expertise in mathematics education. The validation employed an expert validation sheet covering the aspects of material, construction, and language. The experts also provided written comments and suggestions, which served as the basis for revising and refining the prototype. The validation results indicated that the prototype was feasible for use with revisions. The experts also provided written comments and suggestions, which served as the basis for revising and refining the prototype. The validation data were analyzed qualitatively by examining the experts' written comments and suggestions. These qualitative findings guided the revision and refinement of the worked example prototype. The final outcome of this study is a worked example prototype consisting of two worked examples and paired problems designed to facilitate students' mathematical reasoning in the topic of circles.

3. RESULTS AND DISCUSSION

3.1. Results

This study produced a worked example design framework to facilitate students' mathematical reasoning in the topic of circles. The following discussion presents the findings obtained at each stage of the worked example development process.

The analysis phase began with a literature-based needs analysis to identify instructional needs for developing worked examples in learning circle concepts. The analysis indicated that geometry instruction is still predominantly procedural, limiting students' opportunities to recognize relationships among circle concepts, formulate conjectures, and construct mathematical arguments. In addition, students often rely on memorizing formulas rather than understanding the underlying relationships among circle elements. These findings served as the basis for designing worked examples that explicitly integrate mathematical reasoning into each solution step.

Regarding prerequisite knowledge, the literature review indicated that students need prior understanding of points, lines, angles, proportions, and units of length before learning circle concepts. These prerequisite concepts provide the foundation for understanding the relationships among circle elements, including the radius, diameter, circumference, area, arc length, and sector area. Based on this analysis, the worked examples were designed to build on students' prior knowledge by progressively connecting these prerequisite concepts to new circle concepts. Each worked example was therefore structured to guide students from recognizing relationships among circle elements to explaining the reasoning underlying each solution step and constructing mathematical arguments in contextual problem-solving.

The next step involved formulating indicators of mathematical reasoning that serve as the basis for developing the worked examples. These indicators were synthesized from various expert perspectives (NCTM, 2000; Borthwick & Cross, 2018; Nickerson, 2010; Sundstrom, 2016; Lannin et al., 2011). Based on this synthesis, mathematical reasoning is defined as a logical thinking process involving recognizing patterns and relationships, making conjectures and generalizations, and constructing and evaluating arguments. The detailed indicators used in this study are presented in Table 1.

Table 1. Indicators of Mathematical Reasoning

Indicators of Mathematical Reasoning	Activity Description
Recognizing patterns and relationships	Identifying patterns, connections between concepts, or mathematical relationships within a problem.
Making conjectures and generalizations	Formulating conjectures based on identified patterns and generalizing findings logically.
Constructing and evaluating arguments	Developing logical reasoning or proof for a statement and evaluating the validity of emerging mathematical arguments.

Based on the formulated mathematical reasoning indicators and the characteristics of eighth-grade students identified through the literature review, the worked examples were designed to provide structured guidance that gradually supports independent reasoning. Each worked example was organized into sequential solution steps corresponding to the indicators of recognizing patterns and relationships, making conjectures and generalizations, and constructing and evaluating arguments. Furthermore, paired problems were incorporated to encourage students to apply the reasoning demonstrated in the worked examples to similar situations with reduced instructional support. This design was intended to facilitate students' transition from guided reasoning to more independent mathematical reasoning.

The design stage aims to translate the results of the analysis into concrete visual designs and problem structures. The worked example design framework is developed by integrating instructional design principles and Cognitive Load Theory to ensure that information is presented efficiently without overloading students' working memory (Sweller, 2020).

The main instructional strategy applied is the fading guidance principle, which refers to the gradual reduction of instructional support (Renkl, 2014). In this design, learning begins with fully worked-out examples accompanied by conceptual explanations, followed by independent practice problems that have a similar structure (isomorphic problems). This approach allows students to gradually transition from understanding concepts to developing independent problem-solving skills.

The visual design of the problems considers cognitive load management. The split-attention effect is avoided by presenting text and images in an integrated manner, while redundant information is eliminated to ensure that students focus on the core problem. In addition, the visual design emphasizes composite plane figures that require students to perform visual decomposition, thereby supporting the development of spatial ability and geometric reasoning (Sweller et al., 2019).

A key feature of this design is the integration of mathematical reasoning indicators into the solution structure. Each worked example is equipped with stages that encourage students to formulate conjectures and provide logical reasoning (argumentation) for each step taken. Thus, students are not only focused on the final answer but also on the systematic and reflective process of mathematical thinking. To support learning transfer, each worked example is paired with an isomorphic problem that varies in context or values while maintaining the same conceptual structure.

The problems used in the worked examples in this study were adapted from a junior secondary mathematics textbook published Kementerian Pendidikan dan Kebudayaan (2017). However, the adaptation process was not carried out directly but through structural and cognitive transformation. In the original problems, tasks generally emphasize procedural solutions, whereas in this study, the problems were reconstructed to facilitate students' mathematical reasoning through the addition of thinking stages and explicit instructions.

In the original problems, tasks tend to focus on procedural solutions. In this study, the problems were reconstructed by incorporating thinking stages aligned with mathematical reasoning indicators, namely recognizing patterns, making conjectures, and constructing and evaluating arguments.

The transformation was carried out through three main steps: (1) restructuring problems into composite figure analysis-based tasks, (2) adding guiding questions to encourage the formulation of conjectures, and (3) embedding explicit instructions in the form of self-explanation prompts to encourage students to explain the reasoning behind each step of the solution (Renkl, 2014).

Furthermore, the problem design also considers the principles of Cognitive Load Theory to ensure that students' cognitive load remains optimal during the problem-solving process (Sweller, 2020). Thus, the resulting problems not only require final answers but also facilitate students' mathematical thinking processes in a systematic and reflective manner. The differences between textbook problems and the designed problems in this study are presented in Table 2.

Table 2. Comparison between Textbook Problems and Designed Problems Based on Mathematical Reasoning

Aspect	Textbook Problems	Designed Problems
Objectiv	Calculating Results	Developing Reasoning
Structure	Direct	Gradual
Cognitive Activity	Procedural	Analytical and Refle
Output	Final Answer	Mathematical Argumen

Based on table 2, the transformation of problems occurs not only in terms of the presentation structure but also in the learning objectives and the expected cognitive activities of students. The designed problems are intended to encourage mathematical reasoning processes through systematic stages of thinking, enabling students not only to obtain final answers but also to construct and evaluate mathematical arguments.

The initial worked example prototype was evaluated by two mathematics education lecturers from Universitas Negeri Yogyakarta (UNY), both of whom have expertise in mathematics education. The validation employed an expert validation sheet covering the aspects of material, construction, and language. The validation results indicated that revisions were primarily needed in the aspects of material clarity, visual representation, and instructional wording. A summary of the validators' comments, validation decisions, and the corresponding revisions is presented in Table 3.

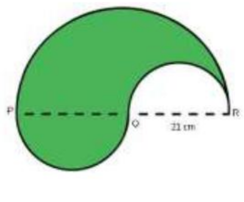
Table 3. Expert Validation Results

Validator	Validation Aspects	Suggestions and Comments	Revisions Made	Validation Decision
Validator 1	• Material, Construction, Language	• Some information in the problems is unclear and needs clarification. • Visual representations need improvement to avoid ambiguity. • Instructional sentences need to be refined.	• Clarified the problem information. • Improved the visual representations to eliminate ambiguity. • Refined the instructional wording for greater clarity.	• Feasible for use with revisions
Validator 2	• Material, Construction, Language	• Visual representations need improvement to avoid ambiguity. • Minor typographical errors were identified in the problem statement and figures.	• Revised the visual representations to eliminate ambiguity • Corrected the typographical errors and ensured consistency between the problem statement and the figures	• Feasible for use with revisions

The suggestions and feedback provided by the validators were used to revise and refine the developed worked example prototype. The revisions focused on improving the clarity of problem information, refining instructional wording, correcting typographical errors, and enhancing visual representations to eliminate ambiguity. Based on the expert validation results, the worked example prototype was considered feasible for use with revisions. Following the revision process, the final prototype consisted of two worked examples accompanied by paired problems that progressively guide students from recognizing patterns and relationships to constructing and evaluating mathematical arguments.

Worked Example 1 was designed with the learning objective of enabling students to determine the circumference of a circle through mathematical reasoning processes. The figure in the problem consists of several interconnected parts of circles, forming a shaded region that is interesting to analyze. This problem was designed to train students to develop their mathematical reasoning skills through activities such as identifying relationships among parts of the circles, making conjectures about the combined circumference, and constructing logical arguments based on the concept of circle circumference.

Thus, this first worked example serves as a bridge between conceptual understanding and procedural application in the context of circle geometry. An example of the worked example design produced during the development stage can be seen in Figure 1.



Observe the green plane figure shown alongside. The figure is bounded by curved lines, each of which is a semicircle. Points P, Q, and R lie on the same horizontal straight line, with Q located exactly midway between P and R. It is given that the length of QR = 21 cm. *(Use $\pi = \frac{22}{7}$).*

a. Which arc segments form the boundary (perimeter) of the figure?
b. Before calculating, write your conjecture about the relationship between the arcs of the large circle and the small circles.
c. Based on your conjecture, calculate the perimeter of the shaded region and provide justification for each step.

Solution Steps

a. Recognizing Patterns and Relationships

The figure consists of:

- One-half of a larger circle
- Two semicircles of smaller circles facing each other
- The diameter of the small circle is 21 cm, which is equal to the radius of the large circle

b. Formulating Conjectures and Generalizations

The relationship between the arcs of the large circle and the small circles is as follows: The perimeter of the shaded region is formed by the combination of the arc of a half-circle and one complete small circle.

Thus, the total perimeter is given by:

Total Perimeter: $\frac{1}{2} \times$ (Circumference of the large circle) $+$ (Circumference of the small circle)

c. Constructing and Evaluating Arguments

Circumference of the large circle (half): $\frac{1}{2} \times 2\pi r = \frac{1}{2} \times 2 \times \frac{22}{7} \times 21 = 66\ cm$

Circumference of the small circle: $\pi d = \frac{22}{7} \times 21 = 66\ cm$

Total Peimeter: $66\ cm + 66\ cm = 132\ cm$

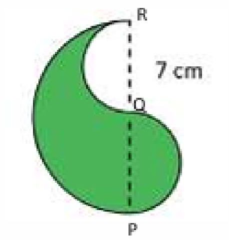
Therefore, the perimeter of the shaded region is 132 cm.

Mathematical Justification:

- Two semicircles of the smaller circles form one complete circle; therefore, it is counted once.
- The entire boundary consists of curved arcs; thus, the total perimeter is obtained by summing the lengths of these arcs.
- The choice of $\pi = \frac{22}{7}$ is based on the diameter being a multiple of 7, which simplifies the calculation.

Figure 1. Worked Example 1

After the worked example was developed, the next step was to construct a paired problem with a similar structure and context, but with variations in numerical values or specific conditions. This problem aims to assess the extent to which students are able to apply the concepts and procedures learned from the previous worked example independently, without direct guidance. An example of the paired problem developed based on Worked Example 1 is presented in Figure 2.



Observe the green plane figure shown alongside. The figure is bounded by curved lines, each of which is a semicircle. Points P, Q, and R lie on the same vertical straight line, with Q located exactly midway between P and R. It is given that the length of QR = 7 cm. *(Use $\pi = \frac{22}{7}$).*

a. Which arc segments form the boundary (perimeter) of the figure?
b. Before calculating, write your conjecture about the relationship between the arcs of the large circle and the small circles.
c. Based on your conjecture, calculate the perimeter of the shaded region and provide justification for each step.

Figure 2. Paired Problem for Worked Example 1

The expected solution to the given problem is structured systematically to illustrate students' mathematical thinking processes. Each step of the solution demonstrates how students understand the relationship between the circumference and diameter of a circle, perform calculations based on the problem context, and draw logical conclusions. Through this process, students are expected to focus not only on the final answer but also on the underlying reasoning. A complete illustration of the solution is presented in Figure 3.

Solution Steps

a. Recognizing Patterns and Relationships

The figure consists of:

- One-half of a larger circle
- Two semicircles of smaller circles facing each other
- The diameter of the small circle is 7 cm, which is equal to the radius of the large circle

b. Formulating Conjectures and Generalizations

The relationship between the arcs of the large circle and the small circles is as follows: The perimeter of the shaded region is formed by the combination of the arc of a half-circle and one complete small circle.

Thus, the total perimeter is given by:

Total Perimeter: $\frac{1}{2} \times$ (Circumference of the large circle) + (Circumference of the small circle)

c. Constructing and Evaluating Arguments

Circumference of the large circle (half): $\frac{1}{2} \times 2\pi r = \frac{1}{2} \times 2 \times \frac{22}{7} \times 7 = 22\ cm$

Circumference of the small circle: $\pi d = \frac{22}{7} \times 22 = 22\ cm$

Total Peimeter: $22\ cm + 22\ cm = 44\ cm$

Therefore, the perimeter of the shaded region is 44 cm.

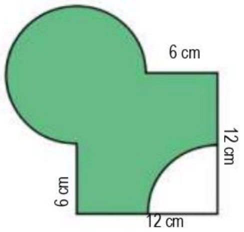
Mathematical Justification:

- Two semicircles of the smaller circles form one complete circle; therefore, it is counted once.
- The entire boundary consists of curved arcs; thus, the total perimeter is obtained by summing the lengths of these arcs.
- The choice of $\pi = \frac{22}{7}$ is based on the diameter being a multiple of 7, which simplifies the calculation.

Figure 3. Solution to the Paired Problem for Worked Example 1

The second worked example was designed with the learning objective of enabling students to determine the area of a circle through mathematical reasoning. At this stage, students are not only required to apply the circle area formula directly, but also to reason about the relationships among different parts of the figure and connect them to the concept of area. The presented figure consists of a composite shape combining a square and parts of circles. Students are asked to systematically explain how to determine the area of the shaded region and to provide mathematical justification for each step of the solution. An example of the worked example 2 design developed in this stage is presented in Figure 4.

Observe the following figure!



The green figure is composed of several connected plane shapes. The side length of the square is 12 cm. Use $\pi = 3.14$. Answer the following questions systematically:

a. Identify the plane figures that make up the shape!
b. Write your conjecture about the appropriate combined formula to determine the area of the shaded region!
c. Calculate the total area and provide mathematical justification for each step!

Solution Steps

a. Recognizing Patterns and Relationships

The figure consists of:

- A square with dimensions 12 cm × 12 cm.
- A $\frac{3}{4}$ portion of a circle added to the upper-left side of the square.
- A $\frac{1}{4}$ portion of a circle at the bottom-right corner that is not shaded (removed).

Given that:

- The side length of the square is 12 cm.
- The radius of the circle is $r = 6$ cm.

b. Formulating Conjectures and Generalizations

The relationship among these parts can be expressed as follows:

Area of the shaded region = Area of the square + $\frac{3}{4}$(area of the circle) − $\frac{1}{4}$(area of the circle)

- Area of the square:

$$s \times s = 12 \times 12 = 144$$

- $\frac{3}{4}$ of the circle's area:

$$\frac{3}{4}\pi r^2 = \frac{3}{4} \times 3.14 \times 6^2 = 84.78$$

- $\frac{1}{4}$ of the circle's area (unshaded part):

$$\frac{1}{4}\pi r^2 = \frac{1}{4} \times 3.14 \times 6^2 = 28.26$$

c. Constructing and Evaluating Arguments

Area of the shaded region $= 144 + 84.78 - 28.26 = 200.52\ \text{cm}^2$

Thus, the area of the shaded region is **200.52 cm²**.

Justification:

- The shaded region is composed of the entire square combined with portions of a circle.
- From the figure, there is a $\frac{3}{4}$ circle added outside the square and a $\frac{1}{4}$ circle removed from one corner.
- Therefore, the total area can be expressed as:

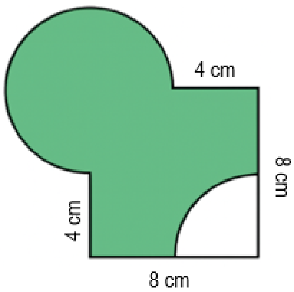
$$\text{Area of the shaded region} = \text{Area of the square} + \frac{3}{4}(\text{circle}) - \frac{1}{4}(\text{circle})$$

- This expression represents adding the shaded circular parts and subtracting the unshaded portion.

Figure 4. Worked Example 2

The problem and solution in Worked Example 2 are designed to guide students in developing a meaningful understanding of the concept of circle area through mathematical reasoning. The solution steps are presented systematically, allowing students to follow a logical line of thinking, starting from identifying shapes and relationships among figures to drawing conclusions about the area of the shaded region. Each step in the worked example is designed to minimize students' cognitive load, in line with the principles of cognitive load theory, so that they can focus more on the reasoning process rather than merely on computational procedures. In addition, a paired problem is provided as a similar exercise to reinforce students' conceptual understanding gained from the previous worked example. An example of the paired problem for Worked Example 2 is presented in Figure 5.

Observe the following figure!



The green figure is composed of several connected plane shapes. The side length of the square is 8 cm. Use $\pi = 3.14$. Answer the following questions in order.
Answer the following questions systematically:

a. Identify the plane figures that make up the shape!
b. Write your conjecture about the appropriate combined formula to determine the area of the shaded region!
c. Calculate the total area and provide mathematical justification for each step!

Figure 5. Paired Problem for Worked Example 2

Although the visual context of the problem retains characteristics from the original source, the structure of the questions and the solution process have been significantly modified. These changes are intended to guide students not only toward obtaining the final answer but also toward engaging in the reasoning process, which involves identifying patterns, formulating conjectures, and providing mathematical justifications at each step of the solution.

3.2. Discussion

The development stage in this study resulted in two worked example designs on circle material that were specifically developed to train junior high school students' mathematical reasoning abilities. The two designs consist of Worked Example 1, which focuses on the concept of circumference, and Worked Example 2, which focuses on the concept of area. The design development was carried out by synthesizing Cognitive Load Theory and instructional design principles to ensure that the presentation of information supports students' thinking processes. According to Sweller, (2020), the presentation of structured worked examples can reduce extraneous cognitive load, allowing students' mental capacity to be allocated to conceptual understanding. This principle is in line with Renkl, (2014), who emphasizes the importance of worked examples as a bridge between declarative and procedural knowledge through the presentation of solution steps accompanied by conceptual reasoning.

Based on expert judgment, the developed worked example prototype is theoretically aligned with the indicators of mathematical reasoning. This alignment is reflected in the organization of problem instructions into three systematic stages of thinking: (a) recognizing patterns and relationships, (b) making conjectures and generalizations, and (c) constructing and evaluating arguments.

First, in the stage of recognizing patterns and relationships (Point a), students are guided to perform visual decomposition. In Worked Example 1, students are guided to recognize the pattern that two small semicircles facing each other form a complete circle. This activity encourages students to identify relationships among geometric elements presented in the figure before performing calculations, as suggested by Borthwick & Cross, (2018), who state that recognizing relationships is a foundational step in developing meaningful mathematical reasoning. Meanwhile, in Worked Example 2, students observe that the shaded region is a combination of a square and parts of circles.

Second, the stage of making conjectures and generalizations (Point b). A distinguishing feature of this design is the explicit instruction for students to formulate conjectures regarding solution strategies before performing calculations. For example, in Worked Example 2, students are guided to construct a mathematical model in which the area of the shaded region is obtained through the addition and subtraction of regions. This process aligns with the views of Sundstrom, (2016); Lannin et al. (2011), who emphasize that the ability to logically generalize observations is a form of higher-order mathematical thinking essential in learning geometry.

Third, the stage of constructing and evaluating arguments (Point c). This design emphasizes that the final answer is not the primary goal. Students are required to complete a “Mathematical Reasoning” column at each step of the solution to justify their decisions (e.g., why a particular part should be subtracted). This is consistent with Lithner, (2017); Mukuka et al. (2023), who argue that genuine reasoning (creative reasoning) emerges when students are able to connect each step of the solution with the underlying conceptual principles, rather than merely following mechanical procedures. In the context of circle area, this argumentative approach responds to the findings of Lehmann, (2024), which state that understanding the area of a circle requires strong conceptual justification, not merely the application of formulas.

In addition to reasoning aspects, this design also considers cognitive aspects. Text and visual information are presented in an integrated format to avoid the split-attention effect, as explained. This approach ensures that students do not need to search for relationships between separate sources of information, thereby minimizing cognitive load and allowing their thinking to focus on the mathematical structures being learned.

Furthermore, the fading guidance strategy is implemented through the presentation of paired problems at the end of each unit. Step-by-step guidance is completely removed in the paired problems, requiring students to transition from understanding examples to independent problem solving. According to Bentley & Yates, (2017), this transition strategy is effective in fostering proportional reasoning and logical reflection in geometry learning contexts, as students are challenged to transfer their understanding to new situations.

Conceptually, the results of the development stage demonstrate that this reasoning-based worked example design is capable of providing meaningful learning experiences. The design guides students to understand relationships among geometric elements, develop the ability to formulate logical conjectures, and test the validity of arguments through mathematical justification. This supports the views of Jablonski & Ludwig, (2023); Weingarden & Buchbinder, (2023), who argue that geometry learning should foster reflective and analytical thinking skills. Therefore, the product developed in this study meets the theoretical criteria as an instructional material model that facilitates students in thinking logically, reflectively, and based on mathematical reasoning.

Compared to textbook problems that generally focus on the direct application of formulas, the problem design in this study demonstrates a paradigm shift toward reasoning-based learning. Students are no longer positioned as mere executors of procedures but as active individuals who construct knowledge through logical and reflective thinking processes. This transformation represents a significant contribution of the study, as it shows that worked examples not only function to reduce cognitive load but can also be designed to stimulate students’ higher-order thinking skills.

Compared with previous studies, the main contribution of this study lies in the integration of mathematical reasoning indicators into the structure of worked examples based on Cognitive Load Theory. Previous studies have mainly emphasized worked examples as a means of reducing cognitive load and improving procedural problem solving (Sweller, 2020; Renkl, 2014). In contrast, the worked examples developed in this study were systematically designed to guide students through recognizing patterns and relationships, making conjectures and generalizations, and constructing and evaluating arguments. This finding extends previous research by demonstrating that worked examples can be designed not only to support procedural learning but also to facilitate mathematical reasoning in geometry learning.

Despite the theoretical contribution of the proposed worked example design, this study has several limitations. The study was limited to the design and development stages and did not include classroom implementation or empirical testing with students. Therefore, the effectiveness of the proposed worked example prototype in improving students' mathematical reasoning, reducing cognitive load, and enhancing learning outcomes has not yet been empirically confirmed. Consequently, the findings of this study should be interpreted as evidence of the theoretical feasibility of the proposed instructional design rather than empirical evidence of its instructional effectiveness. These methodological limitations also limit the generalizability of the findings to broader educational contexts. Future studies are therefore recommended to implement and evaluate the proposed prototype in authentic classroom settings across different mathematical topics and educational levels.

Despite these limitations, the proposed worked example design provides a reference for mathematics teachers and instructional designers in developing reasoning-oriented instructional materials. Rather than presenting solution procedures alone, the design explicitly integrates mathematical reasoning indicators into each solution step, encouraging students to understand relationships among concepts and justify their mathematical thinking. Consequently, the proposed design may support the development of geometry learning materials that promote conceptual understanding while effectively managing students' cognitive load.

4. CONCLUSION

This study produced a worked example prototype on circle concepts consisting of two worked examples and paired problems focusing on the concepts of circumference and area. The product was developed through three stages of the ADDIE model, Analysis, Design, and Developmen with an orientation toward developing junior high school students' mathematical reasoning.

The findings of this study contribute theoretically by demonstrating how Cognitive Load Theory can be integrated with mathematical reasoning indicators within the structure of worked examples. The proposed framework extends the role of worked examples beyond procedural guidance by systematically supporting students in recognizing patterns and relationships, making conjectures and generalizations, and constructing and evaluating mathematical arguments. This is reflected in the presentation of integrated information to minimize extraneous cognitive load, the use of systematic thinking stages (identifying patterns, making conjectures, and evaluating arguments), and the implementation of a fading guidance strategy through paired problems to support the transition from guided learning to independent problem solving.

Based on expert validation results, the developed instrument was deemed feasible for use with revisions. Improvements were made in terms of language, clarity of information, and refinement of visual representations to avoid ambiguity and misconceptions. Thus, the resulting design meets theoretical feasibility criteria as a learning tool that supports mathematical reasoning.

From a practical perspective, the proposed worked example framework provides an alternative approach for developing reasoning-oriented mathematics instructional materials. The framework offers guidance for integrating mathematical reasoning indicators into worked examples while maintaining effective cognitive load management.

Although this study was limited to the design and development stages, the proposed framework provides a theoretical foundation for future classroom implementation and further research on reasoning-oriented mathematics instruction. Future studies are recommended to evaluate its effectiveness across different mathematical topics and educational levels.

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